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On some properties of the ratio of matrix elements $\left|\frac{V_{ub}}{V_{cb}}\right|$ related to the Fermi momentum (p_f)

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Abstract. The most probable value of the Fermi momentum (p_f) has been suggested in the framework of the statistical model and its dependence on the ratio of the CKM matrix elements $|V_{ub}/V_{cb}|$ has been investigated. It has been suggested that the aforesaid ratio should be considerably enhanced compared to the existing estimates so as to reproduce reasonable values of the Fermi momentum, p_f.

1 Introduction

Recently much attention has been paid on the B-meson decay as it is a potential source of measuring the Cabibbo-Kobayashi-Masakawa (CKM) matrix elements. The rare hadronic B-meson decays described by the process $b \to u$ $(b \to u\ell\nu)$ or charmless B-meson decay is particularly important for the study of the CKM matrix elements. Till now the only experimental method to determine $|V_{ub}|$ is through the end point of the lepton energy spectrum of $b \to u$ decay. The ACCMM model [1] provides an excellent method to determine the end point behaviour of the spectrum of a heavy quark into a massless quark and a lepton which is used to study the $|V_{ub}/V_{cb}|$ where the Fermi momentum (p_f) is a very important free parameter in determining $|V_{ub}/V_{cb}|$. Hwang et al. [2,3], Choi et al. [4] have also investigated the dependence of V_{ub}, V_{cb} on p_f in the framework of the ACCMM model.

In the present paper we have estimated p_f in the framework of the statistical model which enables us to estimate it directly by experimentally observed quantities like the decay constant (f_{B}) and mass (m_{B}) of the B-meson. We have also investigated the dependence of $|V_{ub}/V_{cb}|$ on the Fermi momentum in the context of the ACCMM model. With the input of $|V_{ub}/V_{cb}|$ predicted from different models, p_f has been determined. An analysis of the results suggests that the most plausible value of $|V_{ub}/V_{cb}|$ should be more than the recent estimates and that p_f is $\simeq 0.61\,\mathrm{GeV}$.

2 Estimate of p_f

In the low energy limit of the heavy meson annihilation, the Van-Royen-Weisskoff formula gives the relation between the decay constant f_{B} (for the B-meson) and the ground state wavefunction at the origin $\psi_{\mathrm{B}}(0)$ as,

$$f_{\mathrm{B}}^2 = \frac{12}{m_{\mathrm{B}}}|\psi_{\mathrm{B}}(0)|^2 \tag{1}$$

In the statistical model the ground state wavefunction for meson is given by [5],

$$|\psi(r)|^2 = \frac{315}{64\pi r_{\mathrm{B}}^{9/2}}(r_{\mathrm{B}} - r)^{3/2}\theta(r_{\mathrm{B}} - r) \tag{2a}$$

where θ is a step function.

$$\text{At } r = 0\,, \quad |\psi_{\mathrm{B}}(0)|^2 = 315/64\pi r_{\mathrm{B}}^3 \tag{2b}$$

where r_{B} represents the radius or size parameter of the B-meson.

It is pertinent to recall here that in the statistical model [5], the number density of quark (n_q or antiquarks $n_{\bar{q}}$) also represents the probability density of a meson, i.e. $n_q(r) = |\psi_{\mathrm{B}}(r)|^2$ so that $\overline{n}_q$, the average density of quarks (similar for antiquarks also) becomes

$$\overline{n}_q = \frac{1}{2}|\psi_{\mathrm{B}}(0)|^2 = 315/128\pi r_{\mathrm{B}}^3 \tag{3}$$

Hence the average number density would be related to the corresponding (average) Fermi momentum [6] by,

$$p_f^3/3\pi^2 = \overline{n}_q$$

$$\text{or}\,, \quad p_f^3 = \frac{1}{2}|\psi_{\mathrm{B}}(0)|^2\; 3\pi^2 \tag{4}$$

Combining (1), (3) and (4) we get,

$$p_f^3 = 3\pi^2 f_{\mathrm{B}}^2 m_{\mathrm{B}}/24 \tag{5a}$$

Table 1. Present estimates of $r_{\rm B}$ and p_f corresponding to different potentials [8]

Potentials [8]		$r_{\rm B}$ (GeV^{-1})	p_f (GeV)
1.	$-\frac{\alpha_s}{r}+kr$	2.41	1.18
2.	$A+{\rm B}r^{\alpha}$	3.7	0.77
3.	$\lambda\frac{(r^{\alpha}-1)}{\alpha}+c$	3.7	0.77
4.	$-\frac{\alpha_c}{r}+k^1 r$	3.5	0.81
5.	$c\ln\left(\frac{r}{r_{\rm B}}\right)$	3.65	0.78

and

$$p_f r_{\rm B} = 2\cdot 85 \tag{5b}$$

with $f_{\rm B} = (0.19\pm 0.04)$ GeV [7] and $m_{\rm B} = 5.28$ GeV [3], we get $p_f = 0.61$ GeV from (5a) which is in good agreement with the recent estimates of Hwang et al. ($p_f = 0.54^{+0.16}_{-0.15}$) GeV [3] and Choi et al. [4] ($p_f = 0.68$ GeV).

The expression (5b) represents a relation between Fermi momentum and the radius parameter of the B-meson and it measures the Fermi momentum of the two body bound state. In order to calculate p_f from (5b), the knowledge of $r_{\rm B}$ is essential. We have estimated $r_{\rm B}$ from the relativistic Hamiltonian of heavy meson by minimizing it. The values of p_f are thus obtained with the input of different $r_{\rm B}$ corresponding to different potentials [8]. The results are displayed in Table 1. It is observed that $r_{\rm B} = 0.74$ fm (3.7 GeV) corresponding to $p_f = 0.77$ GeV agrees fairly well with the estimates of [3,4].

3 Determination of p_f (ACCMM model)

The rare decay of B-meson transition via $b\to u\ell\nu$ contributes to the end point energy of lepton spectrum. It has been explicitly shown that the $b\to u\ell\nu$ transition is responsible for the excess of leptons with momenta above the kinematical limit for $b\to e\ell\nu$ transition. In the ACCMM model the absolute value of $|V_{ub}|$ is determined from the behaviour of spectrum near the end point and $|V_{ub}/V_{cb}|$ is expressed as a function of the Fermi momentum p_f.

In the ACCMM [1] model the heavy quark (b) is treated as a virtual particle of invariant mass 'W' such that (with antiquark as the spectator),

$$W^2 = M_{\rm B}^2 + m_{sp}^2 - 2m_{\rm B}\sqrt{p^2+m_{sp}^2} \tag{6}$$

where $M_{\rm B}$ is the mass of the B-meson. 'm_{sp}' represents the mass of the spectator antiquark and 'p' is the momentum of the 'b' quark inside B-meson.

Assuming the momentum distribution of the virtual 'b' quark inside the meson to be of Gaussian type we have,

$$\psi_{\rm B}(p) = \frac{4}{\sqrt{\pi}p_f^3}{\rm e}^{-p^2/p_f^2} \tag{7}$$

The lepton energy spectrum of the B-meson decay is given by,

$$d\Gamma_{\rm B}/dE_l = \int_0^{p_f} dp\; p^2\psi(p)\quad d\Gamma_b/dE_l \tag{8}$$

Table 2. Estimate of p_f for different values of $|V_{ub}/V_{cb}|$

Ref.	$\lvert V_{ub}/V_{cb}\rvert$	p_f (GeV)
ACCMM [1]	0.1	0.297
ARGUS [12]	0.08 ± 0.02	0.326
Melikov [11]	0.108 ± 0.02	0.319
Wirbel [10]	0.3	0.870
Present	0.2	0.61

For the evaluation of $(d\Gamma_{\rm B}/dE_l)$, we have parameterized $d\Gamma_b/dE_l$ by, (for large 'p'),

$$d\Gamma_b/dE_l = A/p^2 + {\rm B} \tag{9}$$

where $A = 0.285$, $B = 0.049$.

From (8) and (9) we get,

$$d\Gamma_{\rm B}/dE_l = \frac{1.23}{p_f^2} - 0.085 \tag{10}$$

The decay width $\tilde{\Gamma}(p_f)$ has been estimated at the end point spectrum of the b-meson decay in the range $2.3 < E_l < 2.6$ where the contribution is due to the rare 'b' decay only so that defining $\tilde{\Gamma}(p_f)$ by,

$$\tilde{\Gamma}(p_f) = \int_{2.3}^{2.6}\frac{d\Gamma_{\rm B}}{dE_l}\cdot E_l$$

we get,

$$\tilde{\Gamma}(p_f) = \frac{0.369}{p_f^2} - 0.025 \tag{11}$$

The experimentally measured width $\Gamma_{\rm expt}$ is given by,

$$\Gamma_{\rm expt} = |V_{ub}|^2\cdot\tilde{\Gamma}(p_f) \tag{12}$$

As $\Gamma_{\rm total}$ is proportional to $|V_{ub}/V_{cb}|^2$, hence we arrived at,

$$\Gamma_{\rm expt}/\Gamma_{\rm total} \propto \left|\frac{V_{ub}}{V_{cb}}\right|^2\cdot\tilde{\Gamma}(p_f) \tag{13}$$

or,

$$\tilde{\Gamma}(p_f) = |V_{ub}/V_{cb}|^2\cdot\tilde{\Gamma}(p_f)_{pf=0.3}/\left|\frac{V_{ub}}{V_{cb}}\right|^2 \tag{14}$$

We have used $|V_{ub}/V_{cb}|^2$ from Isgur [9] and $\tilde{\Gamma}(p_f)$ at $p_f = 0.3$ GeV. With the input of $|V_{ub}/V_{cb}|$ suggested in different models we have estimated p_f using (14). The results are displayed in Table 2. It is found that the existing estimates of $|V_{ub}/V_{cb}|$ generate very low values of p_f except the value given by Wirbel et al. [10].

4 Results and discussions

In the present paper we have estimated p_f in the framework of the statistical model. The value of p_f obtained from (5a) with input of $f_{\rm B}$ and $m_{\rm B}$ agrees closely with the recent estimates [2–4]. As the decay constant ($f_{\rm B}$) and $m_{\rm B}$ can be determined with considerable accuracy, the value of

p_f obtained using them ($f_{\mathrm{B}}, m_{\mathrm{B}}$) may not be far from the real value. Moreover (5b) gives a relation between Fermi momentum and radius of the two quark bound states. It is to be noted that the radius corresponding to a power law potential yields a reasonable value of p_f (0.77 GeV). Hwang et al. [2] have also observed that $p_f = 0.5\,\mathrm{GeV}$ corresponds to the radius of the B-meson $r_{\mathrm{B}} = 0.39\,\mathrm{fm}$, which is very low. They have used the Cornell potential to describe the interquark potential. In the present investigation we have found that $r_{\mathrm{B}} = 0.74\,\mathrm{fm}$ (corresponding to the power law type potential) yields reasonable estimate of p_f. So it may be suggested that the interacting potential between the constituent quark is fairly well described by the power law type potential.

Melikov [11] has analysed the semileptonic decays of the heavy meson within the dispersion formulation of the constituent quark model and have obtained $|V_{ub}/V_{cb}| = 0.108 \pm 0.02$. Although this estimate agrees with the recent value predicted by ARGUS ($|V_{ub}/V_{cb}| = 0.08 \pm 0.02$) [12], it produces low values of p_f like the estimate of [1]. On the other hand, Wirbel et al. [10] have assured that the ratio of CKM matrix elements has the bound $|V_{ub}/V_{cb}| \leq 0.3$ and it produces fairly large values of p_f towards its upper limit as is evident from Table 2. With our computed values of p_f ($= 0.61\,\mathrm{GeV}$) we get $|V_{ub}/V_{cb}| = 0.2$. So in the present investigation it has been observed that to generate reasonable values of p_f $|V_{ub}/V_{cb}|$ should have higher values than the existing estimates. We get $|V_{ub}/V_{cb}| = 0.2$ which is much more than the estimates by other workers [2–4]. It seems that the estimate predicted by Wirbel et al. [10] is more plausible (towards upper limit) and is supported by the present work. However it has been pointed out by ARGUS [12] that using only a restricted portion of the spectrum it is very difficult to extract the value of $|V_{ub}/V_{cb}|$. More experimental efforts are needed for a better estimate of $|V_{ub}/V_{cb}|$ which is a very important parameter for the understanding of CP-violation.

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References

1. Altarelli, G., et al.: Nucl. Phys. B **208** (1982), 365
2. Hwang, D.S., et al.: Z. Phys. C **69** (1995), 107
3. Hwang, D.S., et al.: Phys. Rev. D **54**, 9 (1996), 5620
4. Choi, M.T., Kim, J.K.: Phys. Rev. D **53**, 11 (1996), 6670
5. Banerjee, S.N., et al.: Physica Scripta **34** (1986), 314; Physica Scripta **37** (1988), 201
6. Landau, L.D., Lifshitz, E.M.: Statistical Physics, part I, p. 166. Pergamon
7. Rosner, J.L.: Phys. Lett. B **379** (1996), 267
8. Hwang, D.S., Kim, J.K.: Phys. Rev. D **53**, 7 (1996), 3659
9. Isgur, N., et al.: Phys. Rev. D **39** (1989), 799
10. Wirbel, M., et al.: Z. Phys. C **29** (1985), 637
11. Melikhov, D.: Phys. Lett. B **394** (1997), 385
12. ARGUS Collab.: Phys. Rep. **276**, 5, 6 (1996), 252